On the problem of quantum control in infinite dimensions

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Abstract

In the framework of bilinear control of the Schrödinger equation it has been proved that the reachable set has a dense complemement in $\mathcal{S} \cap \mathcal{H}^2$. Hence, in this setting, exact quantum control in infinite dimensions is not possible. On the other hand it is known that there is a simple choice of operators which, when applied to an arbitrary state, generate dense orbits in Hilbert space. Compatibility of these two results is established in this paper and, in particular, it is proved that the closure of the reachable set of bilinear control is dense in $\mathcal{S} \cap \mathcal{H}^2$. The requirements for controllability in infinite dimensions are also related to the properties of the infinite dimensional unitary group.

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1 Introduction

The problem of controllability of quantum systems in finite dimensions has been settled in many papers (see for example [1] [2] [3] [4] [5] [6]). In contrast, for infinite-dimensional quantum systems, a few questions are still open [7] [8].

In the framework of bilinear control

$$i\frac{\partial}{\partial t}\psi\left(x,t\right)=\left(H_0+g\left(t\right)B\right)\psi\left(x,t\right) \tag{1}$$

with $g\left(t\right)\in L^2\left(\left[0,T\right]\right)$ and operators such that H_0 generates a continuous semigroup and B is bounded, Turinici [9] has adapted a result of Ball-Marsden-Slemrod [10] to show that the set of reachable states from any $\psi_0\in\mathcal{S}\cap\mathcal{H}^2$ has a dense complement in $\mathcal{S}\cap\mathcal{H}^2$, $\mathcal{S}$ being the Hilbert sphere and $\mathcal{H}^2$ the $W^{2,2}$ Sobolev space. This is a very general result that applies whenever the operators in (1) generate a piecewise (in time) countable sequence of continuous evolution operators. Then, because continuous maps map compact sets into compact sets, the reachable set is a countable union of compact sets. In infinite-dimensional complete metric spaces, compact sets are nowhere dense hence, by Baire's theorem, the reachable set is a first category set with dense complement. Therefore exact bilinear controllability cannot be achieved in $\mathcal{S}\cap\mathcal{H}^2$.

Several ways may be devised to go beyond this result. Compactness is an internal property of sets but nowhere density is not, it depends on the ambient space. Therefore, for example in a higher regularity space, exact controllability might be achieved. This is the situation in the local controllability results [11] [12] in $\mathcal{S}\cap\mathcal{H}^7$. Another, less explored possibility, would be to choose a control operator B that does not generate a continuous evolution operator.

However, what is really important from the physical point of view, is not exact but approximate controllability, that is, the possibility to approach any target state with arbitrary accuracy. In the bilinear control setting in $\mathcal{S}\cap\mathcal{H}^2$ this would correspond to prove that the reachable set is dense in $\mathcal{S}\cap\mathcal{H}^2$. This is likely to happen because the closure of a first category set is not in general of first category. In fact, the closure of a linear set is of first category if and only if it is itself nowhere dense.

Results on approximate controllability in infinite dimensions already exist in particular cases or imposing some restrictions on the H_0 and B operators

or on their domains ([13] [14] [15] [16] [17]). For example, the exact controllability in the $\mathcal{H}^7$ Sobolev space for a 1D potential, in [11] [12], implies approximate controllability in $\mathcal{L}^2$. In [14] the spectrum is considered to have only finitely many discrete eigenvalues and in [13] the domain must be bounded. In [16] aproximate controllability requires the spectrum of H_0 to be non-resonant and the potential B to couple directly or indirectly every pair of eigenvectors of H_0. However these conditions were later shown [17] to be generic in some sense.

Also, the authors in [18] developed the notion of finitely controllable infinite dimensional systems. They consider a nested set of finite-dimensional subspaces of Hilbert space of which the smallest one is controllable and in each subspace H_α acts a set G_α of operators such that any orbit generated by $\exp(G_\alpha)$ contains a vector in a lower dimensional subspace. Then they prove that any vector in one of the finite-dimensional subspaces may be mapped into any other vector in another finite-dimensional subspace. This is a powerful result with practical applications but is not infinite-dimensional controllability. The subtlety of this difference is related to the fact that G_∞ (Eq.12) is a proper subgroup of the infinite-dimensional unitary group (see Sect.3 for details).

Here we follow a different approach. That approximate controllability is possible in $\mathcal{S}$ had already been proved in [19] by the explicit construction of a small set of unitary operators that, operating in any $\psi_0 \in \mathcal{S}$, reach an arbitrarily small neighborhood of any target state ψ. This result has been later generalized to open quantum systems [20]. However, this does no settle the question of approximate bilinear controllability because it is not obvious that the unitary operators used in [19] can be generated in the setting of Eq.(1).

This is one of purposes of this paper, namely to show that, given any initial and target states (ψ_0, ψ) and an approximation accuracy δ, it is possible to generate by bilinear control the required evolution. Use will be made of the results in Ref.[19], which allows to prove infinite-dimensional controllability with very mild conditions on the free Hamiltonian H_0. Approximations of the shift operator play an important role in this construction. Why the shift operator or some other *essentially infinite dimensional operator* is essential for control in infinite dimensions is related to the properties of the infinite dimensional unitary group. This is explained in detail in Section 3 and an alternative representation of the shift operator is also described. The role of essentially infinite dimension operators in the controllability results may

also shed some light on the nature of the operator conditions used in past attempts to prove approximate controllability in infinite dimensions.

2 Approximate bilinear control in infinite dimensions

The set of operators that in [19] were shown to implement approximate controllability in infinite dimensions are the operators of an $U(2)$ group and the shift operator. By the choice of a countable basis, any separable Hilbert space is shown to be isomorphic to $\ell^2(\mathbb{Z})$, the space of double-infinite square-integrable sequences

$$a = \{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots\} \in \ell^2(\mathbb{Z}) \tag{2}$$

$|a| = \left(\sum_{-\infty}^{\infty} |a_k|^2\right)^{\frac{1}{2}} < \infty$, with basis

$$e_k = \{\cdots, 0, 0, 1_k, 0, 0, \cdots\}$$

It was in this setting that the results in [19] were derived, the shift operator U_+ being

$$U_+ e_k = e_{k+1}, \qquad k \in \mathbb{Z} \tag{3}$$

with inverse

$$U_+^{-1} e_k = e_{k-1}, \qquad k \in \mathbb{Z} \tag{4}$$

and the $U(2)$ group operating in the linear space spanned by e_0 and e_1 and leaving the complementary space unchanged. It was then shown that once an initial and target states (ψ_0, ψ) and an accuracy δ are defined, one may, by the application of these operators go, in a finite number of steps, from ψ_0 to a ψ_n such that $\|\psi - \psi_n\| < \delta$.

In the space of double infinite sequences, one may choose a representation Hilbert space $L^2(0, 2\pi)$, the domain of the operators H_0 and B being

$$D = \left\{\psi \in \mathcal{H}^2; \psi(0) = \psi(2\pi)\right\}$$

Then

$$\left\{e_k = \frac{1}{\sqrt{2\pi}} e^{ik\theta}; k \in \mathbb{Z}\right\}$$

and the shift operator is

$$U_+ = e^{i\theta} \tag{5}$$

Using this background knowledge we now prove:

Proposition: *Approximate bilinear quantum control, with H_0 generating a strongly continuous semigroup and bounded control operators B, is possible in infinite dimensions. That is, the reachable set is dense in $S \cap H^2$.*

The proof proceeds by showing that the $U(2)$ and the shift operators may be approximated with arbitrary precision by bounded operators in the bilinear control context.

Let $U_1, U_2, \cdots, U_n$ be the finite set of $U(2)$ and shift operators that take ψ_0 to a state ψ_n through a sequence of states $\psi_1 = U_1\psi_0, \psi_2 = U_2\psi_1, \cdots \psi_n = U_n\psi_{n-1}$ such that $\|\psi - \psi_n\| < \delta$. Now, considering another set of approximating operators $U_1', U_2', \cdots, U_n'$, and defining $\psi_1' = U_1'\psi_0, \psi_2' = U_2'\psi_1', \cdots \psi_n' = U_n'\psi_{n-1}'$, we have the following estimate

$$\psi_n' - \psi_n = \sum_{k=1}^{n} U_n' \cdots U_{n-k+2}' \left(U_{n-k+1}' - U_{n-k+1} \right) \psi_{n-k}$$

$$\|\psi_n' - \psi_n\| \leq \sum_{k=1}^{n} \left\| \left(U_{n-k+1}' - U_{n-k+1} \right) \psi_{n-k} \right\| \tag{6}$$

The generator θ of the shift operator (5) is not an operator in D but it can be approximated by bounded operators in D. Consider the following family of bounded operators in D

$$B_p(\theta) = \pi - 2\sum_{k=1}^{p} \frac{\sin(k\theta)}{k} \tag{7}$$

and, for an arbitrary normalized state $\phi = \sum_k a_k e_k$ in $L^2(0, 2\pi)$, compute

$$\begin{aligned}
\left\| \left(U_+ - e^{iB_p(\theta)} \right) \phi \right\|^2 &= \frac{1}{2\pi} \left\| \sum_k a_k e^{i(k+1)\theta} - \sum_k a_k e^{i(k\theta + B_p(\theta))} \right\|^2 = \\
&= \frac{1}{\pi} \int_0^{2\pi} d\theta \sum_k \sum_{k'} a_k^* a_{k'} e^{-i(k-k')\theta} \left(1 - \cos\left(B_p(\theta) - \theta\right)\right) \\
&\leq \frac{1}{\pi} \left\{ \int_0^{2\pi} d\theta \left(1 - \cos\left(\sum_{k=p+1}^{\infty} \frac{2}{k} \sin(k\theta) \right) \right) \right\}
\end{aligned}$$

The argument of the cosine in the last term is a Fourier series remainder which, for $\theta \neq 2\pi$, may be made as small as desired by choosing a sufficiently large p. Because the inequality does not depend on ϕ we obtain a norm estimate

$$\left\| U_{+} - e^{iB_p(\theta)} \right\| \leq \varepsilon_{U_{+}}\left(p\right) \tag{8}$$

for arbitrarily small $\varepsilon_{U_{+}}\left(p\right)$.

Now, if H_0 generates a strongly continuous semigroup, $H_0 + g\left(t\right) B_p\left(\theta\right)$ with $B_p\left(\theta\right)$ and $g\left(t\right)$ bounded, is also the generator of a strongly continuous semigroup [22]. Then

$$\left\| \left(e^{iB_p(\theta)} - e^{i\Delta t\left(H_0 + \frac{1}{\Delta t} B_p(\theta)\right)} \right) \psi \right\| \leq \varepsilon_B\left(\Delta t, \psi\right)$$

for any ψ, $\varepsilon_B\left(\Delta t, \psi\right)$ being as small as desired by a sufficiently small choice of Δt.

A similar reasoning applies to a control operator to add to H_0 to approximate the $U\left(2\right)$ transformations to precison $\varepsilon_{U_2}\left(\Delta t, \psi\right)$. Now suppose that to reach ψ_n from the initial state ψ_0 one needs L applications of the shift operator and N $U\left(2\right)$ transformations. Then choosing $\varepsilon_{U_{+}}\left(p\right)$, $\varepsilon_B\left(\Delta t, \psi\right)$ and $\varepsilon_{U_2}\left(\Delta t, \psi\right)$ such that

$$L\left(\varepsilon_{U_{+}}\left(p\right) + \varepsilon_B\left(\Delta t, \psi\right)\right) + N\varepsilon_{U_2}\left(\Delta t, \psi\right) \leq \delta$$

one concludes from (6) that the desired control precision is obtained. This completes the proof.

3 The shift operator and the infinite dimensional unitary group

In the proof of approximate controllability in infinite dimensions in [19], the shift operator played an important role. Of course, the choice of operators implementing quantum control in infinite dimensions is not unique, but the fact that an operator with properties similar to the shift is needed reflects the special features of the infinite-dimensional unitary group. The infinite dimensional unitary and orthogonal groups, $U\left(\infty\right)$ and $O\left(\infty\right)$, are clearly transitive in complex and real infinite-dimensional Hilbert space. Therefore the operators that implement control in infinite dimensions must somehow

be able to generate these groups. The suitable mathematical setting for the groups $U(\infty)$ or $O(\infty)$ is a Gelfand triplet

$$E^* \supset L^2\left(\mathbb{R}^d\right) \supset E \tag{9}$$

E being a nuclear space obtained as the limit of a sequence of Hilbert spaces with successively larger norms. An element g of $U(\infty)$ is a transformation in E such that

$$\|g\xi\| = \|\xi\| \tag{10}$$

By duality $\langle x, g\xi\rangle = \langle g^* x, \xi\rangle$, $x \in E^*, \xi \in E$, the infinite-dimensional unitary group is also defined on E^*, the two groups being algebraically isomorphic.

For the harmonic analysis on $U(\infty)$ one needs functionals on E^*. $U(\infty)$ is a complexification of $O(\infty)$ and a standard result states that if a measure μ is invariant under $O(\infty)$ it must be of the form

$$\mu = a\delta_0 + \int \mu_\sigma dm(\sigma)$$

a sum of a delta and Gaussian measures μ_σ with variance σ^2. Hence we are led to consider the (L^2) space of functionals on E^* with a $O(\infty)$ −invariant Gaussian measure

$$\left(L^2\right) = L^2(E^*, B, \mu)$$

B being generated by the cylinder sets in E^* and μ the measure with characteristic functional

$$C(f) = \int_{S^*} e^{i\langle x,f\rangle} d\mu(x) = e^{-\frac{1}{2}\|f\|^2}, \qquad x \in E^*, f \in E$$

In conclusion: the proper framework to study transitive actions and functional analysis in infinite dimensional quantum spaces is the complex white noise setting [21]. In this context many useful results are already available. For example, the regular representation of $U(\infty)$

$$U_g\varphi(z) = \varphi(g^* z), \qquad z \in E_c^*, \varphi \in \left(L_c^2\right) \cong \left(L^2\right) \otimes \left(L^2\right)$$

splits into irreducible representations [23] corresponding to the Fock space (chaos expansion) decomposition of (L_c^2)

$$\left(L^2\right) = \oplus_{n=0}^{\infty}\left(\oplus_{k=0}^{n} H_{n-k,k}\right)$$

$H_{n-k,k}$ being a complex Fourier-Hermite polynomial of degree $(n-k)$ in $\langle z, \xi \rangle$ and of degree k in $\left\langle \bar{z}, \bar{\xi} \right\rangle$

Furthermore, results concerning a classification of the subgroups of $U(\infty)$ are useful to understand the requirements of quantum control in infinite dimensions. In particular one must distinguish between subgroups that only involve transformations that may be approximated by finite-dimensional transformations like G_∞, obtained as the limit of a sequence of finite-dimensional unitary groups

$$G_n = \left\{ g \in U(\infty),\, g|_{V_n} \in U(n),\, g|_{V_n^\perp} = I \right\} \tag{11}$$

$$G_\infty = \text{proj} \lim_{n\to\infty} G_n \tag{12}$$

from those that contain transformations changing, in a significant way, infinitely many coordinates. These group elements are called *essentially infinite-dimensional* (see the section 4 for a definition). The essential point to remember is that to generate $U(\infty)$, and therefore to be transitive in infinite dimensions, some essentially infinite dimensional elements are needed. This is why the shift operator or some other essentially infinite-dimensional operation is required for control in $\mathcal{S} \cap \mathcal{H}^2$.

In our mathematical construction we have represented a separable Hilbert as a space of double-infinite sequences. Given the importance of essentially infinite-dimensional operators for the quantum control in $\mathcal{S}\cap\mathcal{H}^2$ we include in the next section an implementation of the shift operator in an oscillator-like basis, which may be closer to the usual physical applications.

4 The shift operator in an oscillator-like basis

In the Gelfand triplet setting

$$E \subset H \subset E^*$$

with the white noise measure μ in E^*, choose an orthonormal basis in E

$$\{\xi_i : i = 0, 1, 2, \cdots\}$$

In this basis one has the usual raising and lowering operators a^+ and a and define the operators

$$\begin{array}{rcl} A^+ & : & A^+\xi_i = \xi_{i+1} \\ A & : & \begin{array}{l} A\xi_i = \xi_{i-1}; i \neq 0 \\ A\xi_0 = 0 \end{array} \end{array}$$

that is, $A^+ = a^+ \frac{1}{\sqrt{a^+a+1}}$ and $A = \frac{1}{\sqrt{a^+a+1}}a$.

The projection operator P_0 on the basis state ξ_0 is

$$P_0 = |\xi_0\rangle \langle\xi_0| = 1 - A^+A$$

and in any other state ξ_n is

$$P_n = |\xi_n\rangle \langle\xi_n| = \left(A^+\right)^n P_0 \left(A\right)^n$$

The elementary operator P_{jk} that transforms ξ_k into ξ_j is

$$P_{jk} = |\xi_j\rangle \langle\xi_k| = \left(A^+\right)^j P_0 \left(A\right)^k$$

and one also define the following parity operators

$$P_\pm = \frac{1}{2}\left(1 \pm e^{i\pi a^+ a}\right)$$

Now the operator

$$U_+ = \left(A^+\right)^2 P_+ + \left(A\right)^2 P_- + P_{01}$$

plays the same role as the shift operator in the space of double-infinite square-integrable sequences, as may easily be seen by the appropriate renumbering of a double infinite sequence. Notice that this operator is different from the resonant driving field $(u\,(t)\,x)$, used in the discussions of controllability of the harmonic oscillator [24] [25], which together with free Hamiltonian generates a four-dimensional Lie algebra. U_+ together with an $SU\,(2)$ group acting in the subspace $\{\xi_0, \xi_2\}$ generates an infinite-dimensional Lie algebra and controllability in infinite dimensions is obtained. U_+ is an essentially infinite dimensional operator. This notion is rigorously defined through the *average power* [21], that is,

$$ap\,(U_+)\,(x) = \lim \sup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x, U_+\xi_n - \xi_n\rangle^2$$

$x \in E^*$. If $ap(U)(x)$ for an operator U is positive almost surely for the measure μ in E^*, the operator is called *essentially infinite-dimensional*. Qualitatively it means that it acts, in a significant way, in infinitely many coordinates. In the opposite case, if $ap(U)(x) = 0$ almost surely, then U may be approximated by transformations acting on finite dimensional subspaces. The average power of U_+ is 2 almost surely.

Other essentially infinite-dimensional operators may be obtained by constructions similar to the one used for U_+. As follows from the nature of the infinite-dimensional unitary group, at least one such operator (or an arbitrarily close approximation there of) is needed to obtain density in $\mathcal{S} \cap \mathcal{H}^2$.

5 Conclusions

In addition to establishingh that under mild conditions the closure of the reachable set of bilinear control is dense in $\mathcal{S} \cap \mathcal{H}^2$, we have also put in evidence the special role of essentially infinite-dimensional operators in quantum control.

The central role here was played by the shift operator and approximations thereof. This is an operator that behaves like the application of a magnetic field pulse to a charged particle in a circle (a charged plane rotator), which shifts the eigenstates one level up. Other simple essentially infinite-dimensional operators are described in Ref.[21], which may be used as a guide to develop control methodologies for infinite-dimensional quantum systems.

References

[1] G. M. Huang, T. J. Tarn and J. W. Clark; *On the controllability of quantum-mechanical systems*, J. Math. Phys. 24 (1983) 2608-2618.

[2] V. Ramakrishna, M. Salapaka, M. Dahleh, H. Rabitz and A. Peirce; *Controllability of molecular systems*, Phys. Rev. A51 (1995) 960-966.

[3] S. G. Schirmer, H. Fu and A. I. Solomon; *Complete controllability of quantum systems*, Phys. Rev.A 63 (2001) 063410.

[4] C. Altafini; *Controllability of quantum mechanical systems by root space decomposition of* $SU(n)$, J. Math. Phys. 43 (2002) 2051–2062.

[5] G. Turinici and H. Rabitz; *Wavefunction controllability in quantum systems*, J. Phys. A36 (2003) 2565-2576.

[6] F. Albertini and D. D'Alessandro; *Notions of controllability for bilinear multilevel quantum systems*, IEEE Trans. on Automatic Control 48 (2003) 1399-1403.

[7] R. Illner, H. Lange and H. Teismann; *Limitations on the control of Schrödinger equations*, ESAIM: Control, Optimisation and Calculus of Variations 12 (2006) 615-635.

[8] R.-B. Wu, T.-J. Tarn and C.-W. Li; *Smooth controllability of infinite-dimensional quantum mechanical systems*, Phys. Rev. A 73 (2006) 012719.

[9] G. Turinici; *Analyse de méthodes numériques de simulation et contrôle en chimie quantique*, p. 106, Ph. D. Thesis, Univ. Pierre et Marie Curie, Paris 2000.

[10] J. M. Ball, J. E. Marsden and M. Slemrod; *Controllability for distributed bilinear systems*, SIAM J. Control and Optimization 20 (1982) 575-597.

[11] K. Beauchard; *Local controllability of a 1-D Schrödinger equation*, J. Math. Pures et Appl. 84 (2005) 851-956.

[12] K. Beauchard and J.-M. Coron; *Controllability of a quantum particle in a moving potential well*, J. Funct. Anal. 232 (2006) 328-389.

[13] Y. Nersesyan; *Growth of Sobolev norms and controllability of the Schrödinger equation*, Comm. Math. Phys. 290 (2009) 371-384.

[14] M. Mirrahimi; *Lyapunov control of a quantum particle in a decaying potential*, Ann. Inst. H. Poincaré, Anal. Non Linéaire 26 (2009) 1743-1765.

[15] S. Ervedoza and J.-P. Puel; *Approximate controllability for a system of Schrödinger equations modeling a single trapped ion*, Ann. Inst. H. Poincaré, Anal. Non Linéaire 26 (2009) 2111-2136.

[16] T. Chambrion, P. Mason, M. Sigalotti and U. Boscain; *Controllability of the discrete-spectrum Schrödinger equation driven by an external field*, Ann. Inst. H. Poincaré, Anal. Non Linéaire 26 (2009) 329-349.

[17] P. Mason and M. Sigalotti; *Generic controllability properties for the bilinear Schrödinger equation*, Comm. in Partial Diff. Equations 35 (2010) 685-706.

[18] A. M. Bloch, R. W. Brockett and C. Rangan; *Finite controllability of infinite-dimensional quantum systems*, arXiv:quant-ph/0608075.

[19] W. Karwowski and R. Vilela Mendes; *Quantum control in infinite dimensions*, Phys. Lett. A 322 (2004) 282-285.

[20] R. Vilela Mendes; *Universal families and quantum control in infinite dimensions*, Phys. Lett. A 373 (2009) 2529-2532.

[21] T. Hida and Si Si; *Lectures on white noise functionals*, World Scientific, Singapore 2008.

[22] R. S. Phillips; *Perturbation theory for semi-groups of linear operators*, Trans. Amer. Math. Soc. 74 (1953) 199-221.

[23] K. Okamoto and T. Sakurai; *An analogue of Paley-Wiener theorem for the infinite dimensional unitary group*, Hiroshima Math. J. 12 (1982) 529-541.

[24] K. Kime; *Control of transition probabilities of the quantum mechanical harmonic oscillator*, Applied Math. Lett. 6 (1993) 11-15.

[25] M. Mirrahimi and P. Rouchon; *Controllability of quantum harmonic oscillators*, IEEE Trans. on Automatic Control 49 (2004) 745-747.